FIXED POINT THEORY AND BANACH FUNCTION SPACES

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This thesis contains original results in functional analysis. In Chapter 2, Chapter 3, and the appendix minimal invariant sets of certain mappings are characterized. Of particular interest, the family of minimal invariant sets for Alspach's mapping is described in Chapter 2. Also, the theory of Komlós sets is advanced significantly in Chapter 4 adding to the general theory of Banach function spaces.

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PREFACE

There is no more important person in my life than Judy. Her presence and help make it possible for me to be successful. She has done everything from installing Tex on my computer to showing me how to create and include figures in LaTeX.

My advisor, Chris Lennard, is perhaps my favorite person to speak with in Pittsburgh. He has both lead me and walked beside me on my path of mathematical maturity. His ability to discern when to do which is impeccable. Concerning this document, I would like to thank Chris for his help with grammar, wording, typos, and that one mathematical error.

More generally, I would like to thank the three people that affected my mathematical life profoundly. Gary, my dad, began playing games with me very early in life. Perhaps the earliest memory I have (though vague) is of playing checkers repeatedly. I think of proofs as methods of game play that ensure victory. He showed me the power and joy of logic.

Dr. Paul Lewis introduced me to the rigor of logic in a proofs class. He presented the beauty and certainty of logic grounded in axioms in such an enticing manner that I longed to learn more. My fascination with flawless logic persists.

I have learned a lot from my advisor, Chris. Some was taught. Some was caught. Some was mathematical. Some was not. In particular, Chris showed me the importance of self-evaluation. I learned to assess and adjust my mathematical decisions, direction, and focus. This is the skill I prize most from the time I have spent as a graduate student.

1.0 INTRODUCTION

This thesis contains results from two areas of analysis: Fixed point theory and Banach function spaces. Fixed point theory originally aided in the early developement of differential equations. Among other directions, the theory now addresses certain geometric properties of sets and the Banach spaces that contain them. Banach function spaces is a very general class of Banach Spaces including all L_p spaces for $1 \leq p \leq \infty$, Orlicz spaces, and Orlicz-Lorentz spaces as typical examples. The main reason to prove theorems in this setting is the generality. If one isolates the meaningful restrictions on theorems addressing Banach spaces of functions, they can be easily seen in this setting.

1.1 FIXED POINT THEORY

In 1981, Dale Alspach modified the baker transform to produce the first example of a nonexpansive mapping T defined on C that is fixed point free on a weakly compact convex subset of a Banach space [1]. By Zorn's Lemma, there exist minimal weakly compact, convex subsets of C which are invariant under T and fixed point free. In Chapter 2, we provide a complete description of these minimal invariant sets.

Shortly after Alspach's example, Sine presented an example of a nonexpansive mapping S on C of a Banach space that is fixed point free on all of C [1]. Sine's mapping is a composition of mappings involving Alspach's mapping. In Chapter 3, we explore Sine's mapping, find it's unique minimal invariant set, and compare and contrast Alspach's mapping and Sine's mapping.

1.2 BANACH FUNCTION SPACES

Komlós proved that for any sequence $\{f_n\}_n \subset L_1(\mu)$, μ a probability measure, with $\|f_n\| \leq M < \infty$, there exists a subsequence $\{g_n\}_n$ of $\{f_n\}_n$ and a $g \in L_1(\mu)$ such that for any further subsequence $\{h_n\}_n$ of $\{g_n\}_n$

$$\frac{1}{n}\sum_{i=1}^{n} h_n \to g \ a.e.$$

Later, Lennard proved that every convex subset of $L_1(\mu)$ satisfying the conclusion of Komlós' theorem is norm-bounded. Chapter 4 contains generalizations of both theorems to Banach function spaces with certain properties.

2.0 A CHARACTERIZATION OF THE MINIMAL INVARIANT SETS OF ALSPACH'S MAPPING

2.1 BACKGROUND

In 1981, Dale Alspach modified the baker transform to produce an example of a nonexpansive mapping on a weakly compact, convex subset of $L^1[0,1]$ that is fixed point free[1]. There are several examples of non-expansive mappings on weakly compact, convex sets that are fixed point free [4, 11, 12]. Interestingly, each of these mappings involves or resembles Alspach's example. So, Alspach's example remains the typical example of such a pair consisting of a mapping and a set.

Let $(X, \|\cdot\|)$ be a Banach space and $B \subseteq X$. Recall that $U : B \to B$ is said to be *nonexpansive* if

$$\|U(x) - U(y)\| \leq \|x - y\|\,, \text{ for all } x, y \in B.$$

We assume that B is nonempty, bounded, closed and convex. A set $D \subseteq B$ is said to be U-*invariant* if $U(D) \subseteq D$. Nonempty, closed, convex, U-invariant subsets of B are of interest. In particular, a nonempty, closed, convex, U-invariant set $D \subseteq B$ is said to be *minimal invariant* if whenever $A \subseteq D$ is nonempty, closed, convex and U-invariant, it follows that $A = D$. Minimal invariant sets are in this sense the smallest U-invariant subsets of B. Clearly, the singleton containing any fixed point of U is minimal invariant. In this way, minimal invariant sets generalize the concept of fixed points. For more on minimal invariant sets of nonexpansive mappings we refer the reader to [6, 5].

For any nonexpansive fixed point free mapping on a weakly compact, convex set, there exist a minimal invariant subset of positive diameter, by an application of Zorn's Lemma [7].

These minimal invariant sets have not previously been explicitly characterized for Alspach's example or any other such mapping [5, 8]. We will describe all minimal invariant sets of Alspach's mapping, T. The general idea will be to find a formula for T^n. Next, we will show that $\{T^n x\}$ converges weakly for all $x \in C$. Then, we will use [5] to extract all minimal invariant sets.

2.2 PRELIMINARIES

We begin with some definitions. For all integers $n \geq 0$, for all $i \in \{0, \ldots, 2^n - 1\}$, define $E_{(i,n)} := [\frac{i}{2^n}, \frac{i+1}{2^n})$. Also, let

$$C := \{f \in L_1[0,1] : 0 \leq f(x) \leq 1, \, \forall x \in [0,1]\}$$

and

$$S := \left\{ s \in L^1[0,1] : s = \sum_{i=0}^{2^n-1} a_i \, \chi_{E_{(i,n)}} \text{ where } a_i \in \mathbb{R} \text{ and } n \in \mathbb{N} \right\}.$$

Next, for all $\alpha, \beta \in \mathbb{R}, \alpha \wedge \beta := \min\{\alpha, \beta\}$ and $\alpha \vee \beta := \max\{\alpha, \beta\}$. Fix $a, b \in \mathbb{R}$ with $0 \leq a < b$. For all $c \in \mathbb{R}$, we define

$$\operatorname{cut}(a,b,c) := ((a \vee c) \wedge b) - a = (a \vee (c \wedge b)) - a.$$

The restriction on the domain of cut is not necessary, but it simplifies some of our work.

The cut function will be of particular use. So, it will be advantageous to understand it clearly. At its heart, the cut function is a translation of

$$\operatorname{minmax}(a,b,c) := ((a \vee c) \wedge b) = (a \vee (c \wedge b))$$

with the appropriate restrictions on a, b, and c.

Now, we can prove some properties of minmax, and use translations to obtain properties for the cut function. Recall, that $0 \leq a < b$ is assumed in the definition of cut and minmax. First, we note that

$$\operatorname{minmax}(t\,a, t\,b, t\,c) = t \operatorname{minmax}(a,b,c), \, \forall c \in \mathbb{R}\,, \, \forall t > 0\,.$$

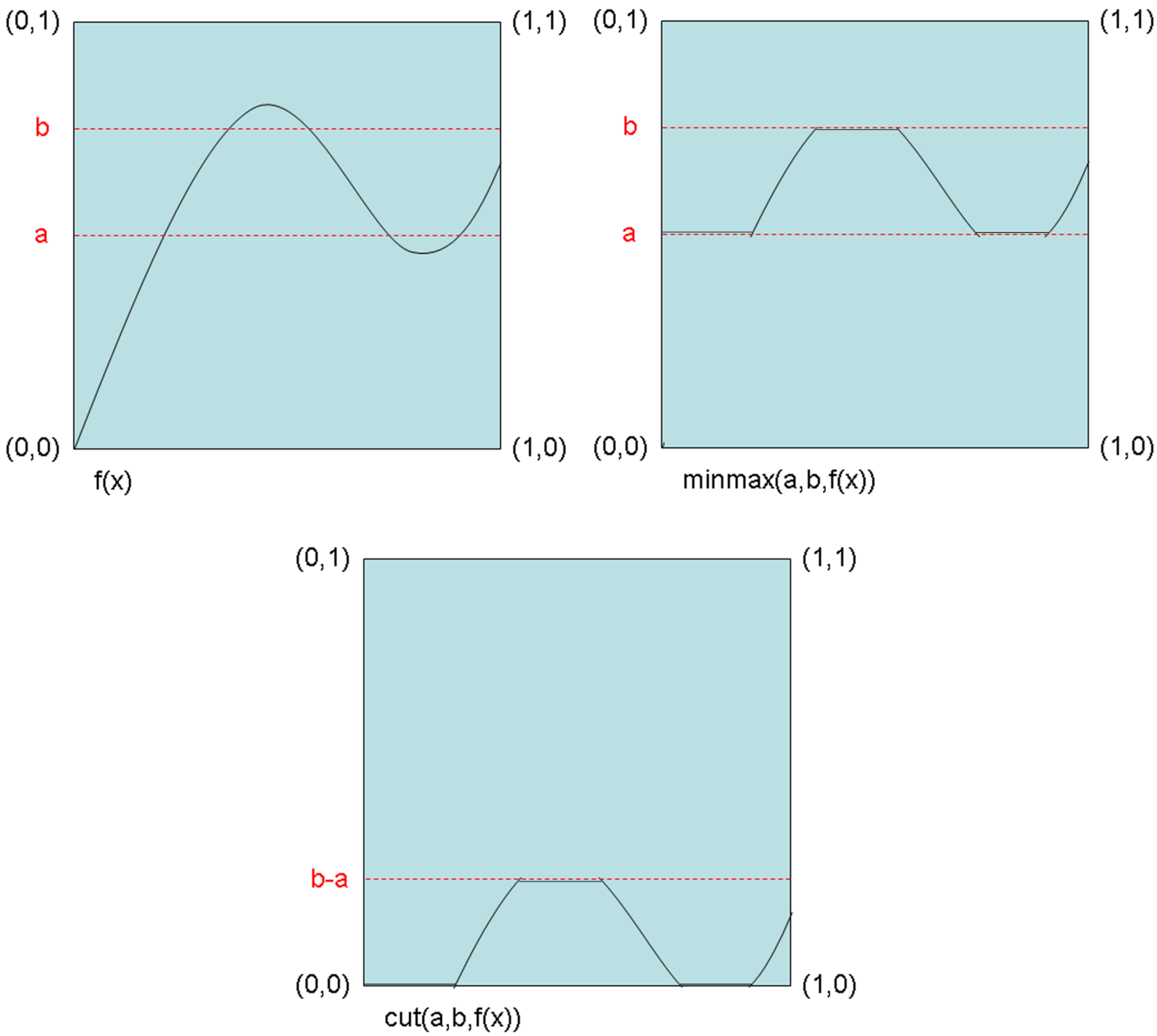

Figure 1: The top left figure represents $y = a$, $y = b$ and a function, f, in C. The top right figure represents $\mathrm{minmax}(a, b, f(x))$. The bottom figure represents $\mathrm{cut}(a, b, f(x))$.

This yields

$$\mathrm{cut}(t\,a, t\,b, t\,c) = t\,\mathrm{cut}(a, b, c),\ \forall c \in \mathbb{R}\,,\ \forall t > 0\,.$$

Second, we can see that $\forall a \leq p < q \leq b\,,\ \forall\, c \in \mathbb{R}$,

$$\begin{aligned}
\mathrm{minmax}(p, q, \mathrm{minmax}(a, b, c)) &= (p \vee (((a \vee c) \wedge b) \wedge q)) \\
&= (p \vee ((a \vee c) \wedge q)) \\
&= (p \vee (a \vee (c \wedge q))) \\
&= (p \vee (c \wedge q)) \\
&= \mathrm{minmax}(p, q, c).
\end{aligned}$$

With a change of variables, we have that $\forall\ 0 \leq p < q \leq b - a\,,\ \forall\, c \in \mathbb{R}$,

$$\mathrm{cut}(p, q, \mathrm{cut}(a, b, c)) = \mathrm{cut}(a + p, a + q, c).$$

Finally, for all real-valued, measurable functions f and g on $[0, 1]$ such that $\mathrm{supp}(f) \cap \mathrm{supp}(g)$ has Lebesgue measure zero,

$$\mathrm{minmax}(a, b, f(x)) + \mathrm{minmax}(a, b, g(x)) = a + \mathrm{minmax}(a, b, f(x) + g(x))\ a.e.$$

and

$$\mathrm{cut}(a, b, f(x) + g(x)) = \mathrm{cut}(a, b, f(x)) + \mathrm{cut}(a, b, g(x))\ a.e.$$

This follows with or without the use of translation by noting that $f(x) \neq 0$ implies $g(x) = 0$ a.e. and that $0 \leq a$. These are the three properties of the cut function that will be used later.

Throughout this chapter, we will extend real-valued, measurable functions f on $[0, 1]$ to $\mathbb{R}$ by defining $f(x) := 0$ for $x \in \mathbb{R}\backslash[0, 1]$. Now, let $T : C \to C$ be defined by

$$(Tf)(x) := \mathrm{cut}\,(0, 1, 2f\,(2x))\,\chi_{E_{(0,1)}}(x) + \mathrm{cut}\,(1, 2, 2f\,(2x - 1))\,\chi_{E_{(1,1)}}(x),$$

for all $x \in [0, 1]$, for all $f \in C$. This is Alspach's mapping [1]. Alspach's mapping is usually written as

$$(Tf)(t) = \begin{cases} 2f(2t) \wedge 1 & , 0 \leq t < \frac{1}{2} \\ (2f(2t-1) \vee 1) - 1 & , \frac{1}{2} \leq t < 1. \end{cases}$$

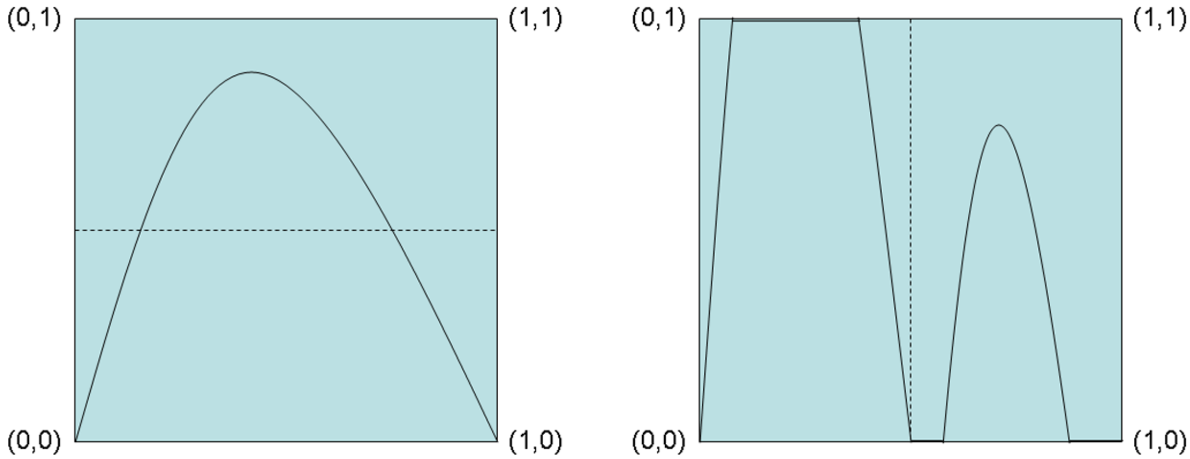

Figure 2: The figure to the left represents a function, f, in C and the line y = 1/2. The figure to the right represents Tf and the line x = 1/2.

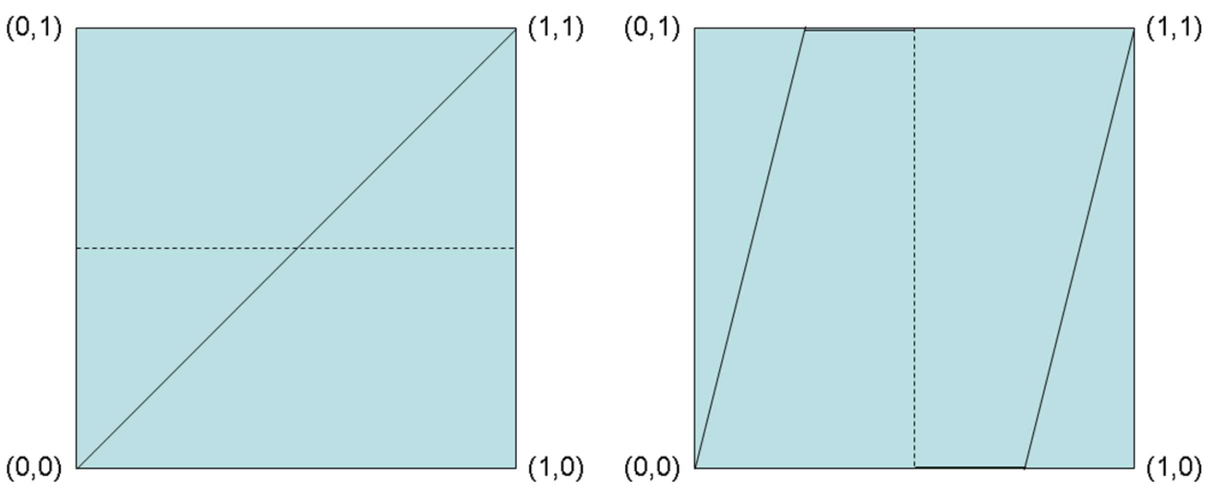

Figure 3: This figure contains another example of T acting on a function in C.

In this chapter, we will often use the identity function x defined by $\mathrm{x}(t) := t$, for all $t \in \mathbb{R}$. Also, we will denote $\int_E f dm = \int_{t\in E} f(t)dt$ by $\int_E f$ for every Lebesgue measurable set E and for every $f \in L^1[0,1]$. Here m is Lebesgue measure.

2.3 ITERATES OF ALSPACH'S MAPPING

Lemma 1. *For every $n \in \mathbb{N}$, for all $f \in C$, for all $x \in [0,1]$, we have*

$$(T^n f)(x) = \sum_{i=0}^{2^n-1} cut\left(\sigma_n(i), \sigma_n(i)+1, 2^n f\left(2^n x - i\right)\right) \chi_{E_{(i,n)}}(x).$$

Here σ_n acts on $i \in \mathbb{N}, 0 \leq i < 2^n$. To define σ, first write $i = \sum_{j=0}^{n-1} d_j 2^j$ with $d_j \in \{0,1\}$ $\forall j$, which is a base 2 representation. Then let $\sigma_n(i) := \sum_{j=0}^{n-1} d_{n-1-j} 2^j$.

Proof. To show that Lemma 1 holds for $n = 1$, it suffices to notice that $\sigma_1(0) = 0$ and $\sigma_1(1) = 1$. Now, to proceed inductively, we will assume that Lemma 1 holds for some fixed $n \in \mathbb{N}$ and show that it holds for $n+1$. We use the three properties of the cut function described above, as well as the fact that $2\sigma_n(i) = \sigma_{n+1}(i)$ and $2\sigma_n(i)+1 = \sigma_{n+1}(i+2^n)$ for $0 \leq i < 2^n$.

It is best to verify these two facts before continuing with the inductions, so that the chain of equalities is not interrupted. For fixed i and n satisfying $0 \leq i < 2^n$, write $i = \sum_{j=0}^{n-1} d_j 2^j =$

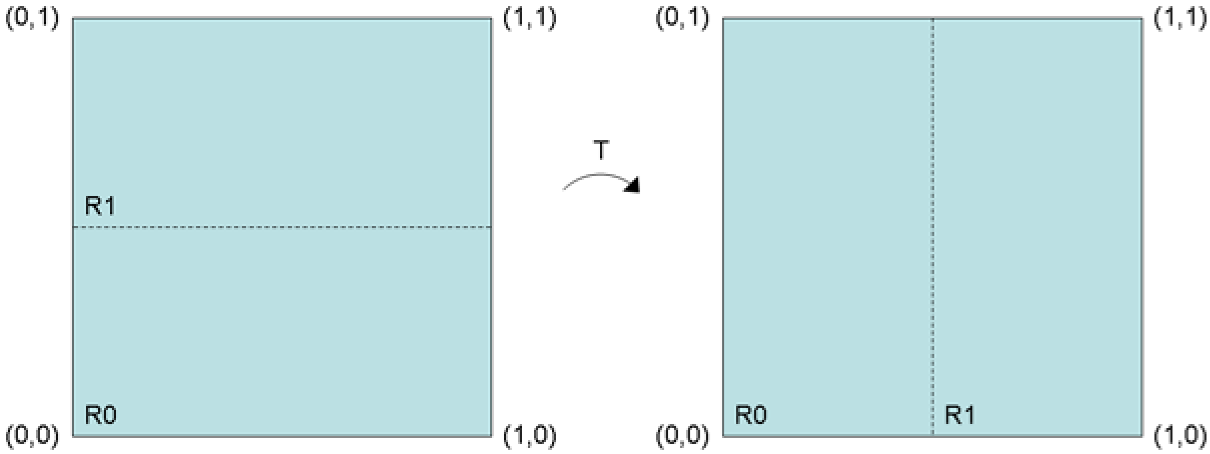

Figure 4: If the graph of a function, f, in C is broken into 2 regions as in the figure to the left, the graph of Tf can be constructed by resizing the portion of the graph of f and translating it to the appropriately labeled position in the figure to the right and adding line segments to ensure that Tf is also in C.

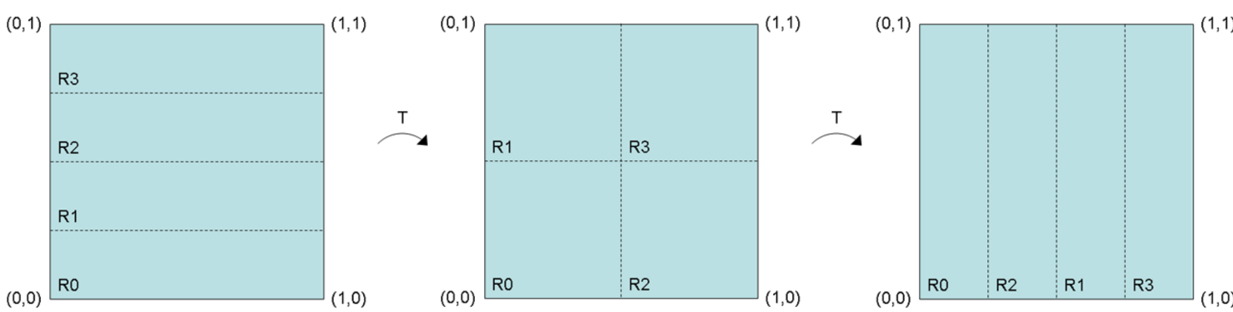

Figure 5: From the left, we can break the graph of any function, f, in C into 4 regions, and see what portion of the graph of Tf is determined by each region of the graph of f(center). The figure to the right, breaks the graph of T^2f into 4 regions that are determined by the 4 identically labeled regions of the graph of f.

$\sum_{j=0}^{n} d_j 2^j$ as above. Note that $d_n = 0$. So,

$$\begin{aligned} 2\sigma_n(i) &= 2\sum_{j=0}^{n-1} d_{n-1-j} 2^j \\ &= \sum_{j=0}^{n-1} d_{n-1-j} 2^{j+1} \\ &= \sum_{k=0}^{n} d_{n-k} 2^k \\ &= \sigma_{n+1}(i). \end{aligned}$$

Also,

$$\begin{aligned} 2\sigma_n(i) + 1 &= \sum_{j=0}^{n-1} d_{n-1-j} 2^{j+1} + 1 \\ &= \sum_{k=1}^{n} d_{n-k} 2^k + 1 \cdot 2^0 \\ &= \sigma_{n+1}\left(\sum_{k=0}^{n-1} d_k 2^k + 1 \cdot 2^n\right) \\ &= \sigma_{n+1}\left(i + 1 \cdot 2^n\right). \end{aligned}$$

Continuing our induction,

$$
\begin{aligned}
T^{n+1}f &= T(T^n f) \\
&= \operatorname{cut}\left(0,1,2\sum_{i=0}^{2^n-1}\operatorname{cut}\left(\sigma_n(i),\sigma_n(i)+1,2^n f\left(2^n(2\mathrm{x})-i\right)\chi_{E_{(i,n+1)}}\right)\right) \\
&\qquad+\operatorname{cut}\left(1,2,2\sum_{i=0}^{2^n-1}\operatorname{cut}\left(\sigma_n(i),\sigma_n(i)+1,2^n f\left(2^n\left(2\mathrm{x}-1\right)-i\right)\chi_{E_{(i+2^n,n+1)}}\right)\right) \\
&= \sum_{i=0}^{2^n-1}\operatorname{cut}\left(0,1,\operatorname{cut}\left(2\sigma_n(i),2\sigma_n(i)+2,2^{n+1}f\left(2^{n+1}\mathrm{x}-i\right)\chi_{E_{(i,n+1)}}\right)\right) \\
&\qquad+\sum_{i=0}^{2^n-1}\operatorname{cut}\left(1,2,\operatorname{cut}\left(2\sigma_n(i),2\sigma_n(i)+2,2^{n+1}f\left(2^{n+1}\mathrm{x}-2^n-i\right)\chi_{E_{(i+2^n,n+1)}}\right)\right) \\
&= \sum_{i=0}^{2^n-1}\operatorname{cut}\left(2\sigma_n(i),2\sigma_n(i)+1,2^{n+1}f\left(2^{n+1}\mathrm{x}-i\right)\chi_{E_{(i,n+1)}}\right) \\
&\qquad+\sum_{i=0}^{2^n-1}\operatorname{cut}\left(2\sigma_n(i)+1,2\sigma_n(i)+2,2^{n+1}f\left(2^{n+1}\mathrm{x}-2^n-i\right)\chi_{E_{(i+2^n,n+1)}}\right) \\
&= \sum_{i=0}^{2^n-1}\operatorname{cut}\left(\sigma_{n+1}(i),\sigma_{n+1}(i)+1,2^{n+1}f\left(2^{n+1}\mathrm{x}-i\right)\chi_{E_{(i,n+1)}}\right) \\
&\qquad+\sum_{i=0}^{2^n-1}\operatorname{cut}\left(\sigma_{n+1}(i+2^n),\sigma_{n+1}(i+2^n)+1,2^{n+1}f\left(2^{n+1}\mathrm{x}-2^n-i\right)\chi_{E_{(i+2^n,n+1)}}\right) \\
&= \sum_{i=0}^{2^{n+1}-1}\operatorname{cut}\left(\sigma_{n+1}(i),\sigma_{n+1}(i)+1,2^{n+1}f\left(2^{n+1}\mathrm{x}-i\right)\right)\chi_{E_{(i,n+1)}}\,.
\end{aligned}
$$

This completes the proof of Lemma 1. ☐

Lemma 2. *For any $f \in C$ and $s \in S$,*

$$\lim_{m\to\infty}\int_{[0,1]} T^m f\cdot s = \|f\|_1\int_{[0,1]} s\ .$$

Proof. Since $s \in S$ is a finite sum of constant functions on intervals of the form $E_{(i,n)}$, it suffices to show Lemma 2 holds for $s = \chi_{E_{(l,n)}}$, where $n \in \mathbb{N}$ and $0 \le l < 2^n$. Fix $m \in \mathbb{N}$. We have that

$$\begin{aligned}
&\int_{[0,1]} T^{n+m} f \chi_{E_{(l,n)}} \\
&\quad = \int_{[0,1]} \left(\sum_{i=0}^{2^{n+m}-1} \operatorname{cut}\left(\sigma_{n+m}(i), \sigma_{n+m}(i)+1, 2^{n+m} f\left(2^{n+m}\mathrm{x} - i\right)\right) \chi_{E_{(i,n+m)}} \right) \chi_{E_{(l,n)}} \\
&\quad = \sum_{i=2^m l}^{2^m l + 2^m - 1} \int_{[0,1]} \operatorname{cut}\left(\sigma_{n+m}(i), \sigma_{n+m}(i)+1, 2^{n+m} f\left(2^{n+m}\mathrm{x} - i\right)\right) \chi_{E_{(i,n+m)}} \\
&\quad = \frac{1}{2^{n+m}} \sum_{i=2^m l}^{2^m l + 2^m - 1} \int_{[0,1]} \operatorname{cut}\left(\sigma_{n+m}(i), \sigma_{n+m}(i)+1, 2^{n+m} f(\mathrm{x})\right).
\end{aligned}$$

From here we wish to reorder the terms in the summation. To that end, for each $2^m l \le i < 2^m l + 2^m$, we can write $i = \sum_{j=0}^{n+m-1} d_j(i) 2^j$; and we see that $d_{j+m}(i) = d_j(l)$, $\forall\, 0 \le j < n$. This implies

$$\begin{aligned}
&\{\sigma_{m+n}(i) : 2^m l \le i < 2^m l + 2^m\} \\
&\qquad = \left\{k : d_j(k) = d_{n-1-j}(l), \forall\, 0 \le j < n, 0 \le k < 2^{m+n}\right\} \\
&\qquad = \{k : k = 2^n v + \sigma_n(l), 0 \le v < 2^m\}.
\end{aligned}$$

Now we can write

$$\int_{[0,1]} T^{n+m} f \chi_{E_{(l,n)}} = \frac{1}{2^{n+m}} \sum_{i=0}^{2^m-1} \int_{[0,1]} \operatorname{cut}\left(2^n i + l, 2^n i + l + 1, 2^{n+m} f(\mathrm{x})\right).$$

It is easy to check that for $c_1, c_2, c_3, c_4 \in \mathbb{R}$ with $c_1 < c_2 \le c_3 < c_4$, and $c_4 - c_3 = c_2 - c_1$, for all real-valued, measurable functions f on $[0,1]$, the following holds:

$$\operatorname{cut}(c_1, c_2, f(\mathrm{x})) \ge \operatorname{cut}(c_3, c_4, f(\mathrm{x})).$$

By applying the above inequality to our nicely ordered sum we obtain

$$\int_{[0,1]} T^{n+m} f \chi_{E_{(l_2,n)}} \le \int_{[0,1]} T^{n+m} f \chi_{E_{(l_1,n)}} \quad , \quad \forall\, 0 \le l_1 < l_2 < 2^n.$$

Using the fact that $\int_{[0,1]} \operatorname{cut}(j, j+1, *) \le 1$, and comparing terms again, we also have

$$\int_{[0,1]} T^{n+m} f \chi_{E_{(0,n)}} \le \int_{[0,1]} T^{n+m} f \chi_{E_{(2^n-1,n)}} + \frac{1}{2^{n+m}}.$$

So, for $0 \le l_1 < l_2 < 2^n$ we have

$$0 \le \int_{[0,1]} T^{n+m} f \chi_{E_{(l_1,n)}} - \int_{[0,1]} T^{n+m} f \chi_{E_{(l_2,n)}} \le \frac{1}{2^{n+m}}.$$

Noticing that $\int_{[0,1]} f = \int_{[0,1]} Tf$ and writing

$$\int_{[0,1]} f = \int_{[0,1]} T^{n+m} f = \sum_{k=0}^{2^n-1} \int_{[0,1]} T^{n+m} f \chi_{E_{(k,n)}} ,$$

it follows that:

$$\begin{aligned}
&\left| \int_{[0,1]} T^{m+n} f \chi_{E_{(l,n)}} - \frac{1}{2^n} \int_{[0,1]} f \right| \\
&\quad = \frac{1}{2^n} \left| 2^n \int_{[0,1]} T^{m+n} f \chi_{E_{(l,n)}} - \sum_{k=0}^{2^n-1} \int_{[0,1]} T^{m+n} f \chi_{E_{(k,n)}} \right| \\
&\quad \le \frac{1}{2^n} \sum_{k=0}^{2^n-1} \left| \int_{[0,1]} T^{m+n} f \chi_{E_{(l,n)}} - \int_{[0,1]} T^{m+n} f \chi_{E_{(k,n)}} \right| \\
&\quad \le \frac{1}{2^n} 2^n \frac{1}{2^{n+m}} = \frac{1}{2^{n+m}} \to 0\,, \text{ as } m \to \infty\,;
\end{aligned}$$

which concludes the proof of Lemma 2. ∎

Theorem 3. *$T^n f$ converges weakly to $\|f\|_1 \chi_{[0,1]}$, $\forall\, f \in C$.*

Proof. Let $f \in C$. Clearly, $f \in L_\infty[0,1]$ and $\|f\|_\infty \le 1$. Take $g \in L_\infty[0,1]$ representing any element of $L_1[0,1]^*$. Note that $g \in L_1[0,1]$. So, there exists a sequence $\{s_m\}_{m=1}^\infty$ in S such that

$$\int_{[0,1]} |s_m - g| \to 0\,, \text{ as } m \to \infty\,.$$

Fix $\epsilon > 0$. Choose $M \in \mathbb{N}$ such that for all $m \ge M$, $\int_{[0,1]} |s_m - g| < \epsilon$. Let $f_n := T^n f$ and $\alpha := \|f\|_1 = \|f_n\|_1$. Then for $m := M$,

$$\begin{aligned}
&\left| \int_{[0,1]} f_n g - \int_{[0,1]} \alpha g \right| \\
&\quad \le \left| \int_{[0,1]} f_n (g - s_m) \right| + \left| \int_{[0,1]} f_n s_m - \int_{[0,1]} \alpha s_m \right| + \left| \int_{[0,1]} \alpha (s_m - g) \right| \\
&\quad \le \|f_n\|_\infty \cdot \|g - s_m\|_1 + \left| \int_{[0,1]} f_n s_m - \alpha \int_{[0,1]} s_m \right| + \alpha \int_{[0,1]} |s_m - g| \\
&\quad \le 2\epsilon + \left| \int_{[0,1]} f_n s_m - \alpha \int_{[0,1]} s_m \right| \le 3\epsilon\,,
\end{aligned}$$

for all n sufficiently large, by Lemma 2. Therefore,

$$\lim_{n\to\infty}\int_{[0,1]} f_n g = \alpha \int_{[0,1]} g\,.$$

This concludes the proof of Theorem 3. □

2.4 INVARIANT SETS

We will now briefly discuss invariant sets. Let (U, B) be the pairing of a mapping U with domain B such that $U : B \to B$, where B is a closed, bounded, convex and nonempty subset of a Banach space. For $f \in B$, define $D_0(f) := \{f\}$, $D_{n+1}(f) := conv\{D_n(f) \cup U(D_n(f))\}$ inductively, and $D_\infty(f) = \overline{\cup_{n=0}^{\infty} D_n(f)}$ [5]. By definition, $D_n(f) \subseteq D_{n+1}(f)$ and $D_n(f)$ is convex. $D_\infty(f)$ is also convex and closed.

A set $M \subseteq B$ is said to be minimal invariant if M is non-empty, closed, convex, U-invariant, and whenever A is a non-empty, closed, convex, U-invariant subset of M, it follows that $A = M$. For all $f \in B$, $D_\infty(f)$ is the smallest closed, convex, U-invariant subset of B that contains f. It is not necessarily minimal invariant though. In fact, for M a minimal invariant set, $f \in M$ if and only if $D_\infty(f) = M$ [6].

Returning to Alspach's map T, let us define $C_\alpha := \{f \in C : \|f\|_1 = \alpha\}$, for all $0 \le \alpha \le 1$. The following lemma is well known and easy to check.

Lemma 4. *Each C_α is T-invariant and $\cup_{\alpha\in[0,1]} C_\alpha = C$.*

Lemma 5. *For every $f \in C_\alpha$, $\alpha\chi_{[0,1]} \in D_\infty(f)$ and $D_\infty\left(\alpha\chi_{[0,1]}\right) \subseteq D_\infty(f)$.*

Proof. Fix any $0 \le \alpha \le 1$ and $f \in C_\alpha$. The sequence $(T^n f)_{n\in\mathbb{N}}$ converges weakly to $\alpha\chi_{[0,1]}$, by Theorem 3. So,

$$\alpha\chi_{[0,1]} \in \overline{conv}\left(\cup_{n\in\mathbb{N}}\{T^n f\}\right) \subseteq D_\infty(f).$$

Since, $D_\infty(f)$ is convex and T-invariant, $D_n(\alpha\chi_{[0,1]}) \subseteq D_\infty(f)$ for every $n \in \mathbb{N}$, and so $D_\infty(\alpha\chi_{[0,1]}) \subseteq D_\infty(f)$ follows. □

Theorem 6. *For every $\alpha \in [0,1], D_\infty\left(\alpha\chi_{[0,1]}\right)$ is the only minimal invariant subset of C_α.*

Proof. Fix $\alpha \in [0,1]$. Suppose M is a minimal invariant subset of C_α. Choose any $f \in M$. From above, we have $M = D_\infty(f)$. Lemma 5 implies $D_\infty\left(\alpha\chi_{[0,1]}\right) \subseteq M$. So, $M = D_\infty\left(\alpha\chi_{[0,1]}\right)$. $\square$

Theorem 7. *For all $\alpha \in (0,1)$, Alspach's mapping T is fixed point free on C_α. Moreover, $\{D_\infty(\alpha\chi_{[0,1]}) : 0 < \alpha < 1\}$ is the collection of all fixed point free minimal invariant subsets of C for T.*

Proof. First, it is worth noting that the singleton containing any fixed point of T must be minimal invariant. Now, assume that T has a fixed point in C_α. Since $D_\infty\left(\alpha\chi_{[0,1]}\right)$ is the only minimal invariant subset of C_α it must be the singleton containing the fixed point. But $T(\alpha\chi_{[0,1]}) \neq \alpha\chi_{[0,1]}$ when $\alpha \in (0,1)$. This is a contradiction.

Further, all minimal invariant subsets of T are contained in some C_α [6]. By Theorem 6, the proof of Theorem 7 is complete. $\square$

3.0 THE UNIQUE MINIMAL INVARIANT SET OF SINE'S MAPPING

3.1 INTRODUCTION

Alspach's mapping is an example of a nonexpansive mapping on a weakly compact, convex subset of $L^1[0,1]$ that is fixed point free. Recall that although we defined Alspach's mapping, T, on

$$C := \{f \in L_1[0,1] : 0 \leq f(x) \leq 1, \, \forall x \in [0,1]\},$$

it is not fixed point free on this set. It is fixed point free on

$$C_{\frac{1}{2}} := \left\{ f \in C : \|f\|_1 = \frac{1}{2} \right\}.$$

In [4] and [12] modified versions of Alspach's mapping are presented and shown to be fixed point free on all of C. Both of these mappings have a unique minimal invariant set in contrast to Alspach's mapping. In this chapter, we will explore Sine's mapping [12].

Sine showed that the mapping we refer to as "Sine's mapping" ($\mathbb{S}$) is fixed point free on C using techniques similar to those found in [1]. The existence of at least one minimal invariant set is obtain easily using Zorn's lemma. That was essentially the extent of knowledge concerning Sine's mapping prior to this work.

Here, we will develop the tools to show that $\frac{1}{2}\chi_{[0,1]} \in D_\infty(f)$ for all $f \in C$. This will give us the existence and uniqueness of a minimal $\mathbb{S}$-invariant subset of C without the use of Zorn's lemma. As with Alspach's mapping, there is an iterative method for constructing from below. We will also give some supersets of the minimal invariant set.

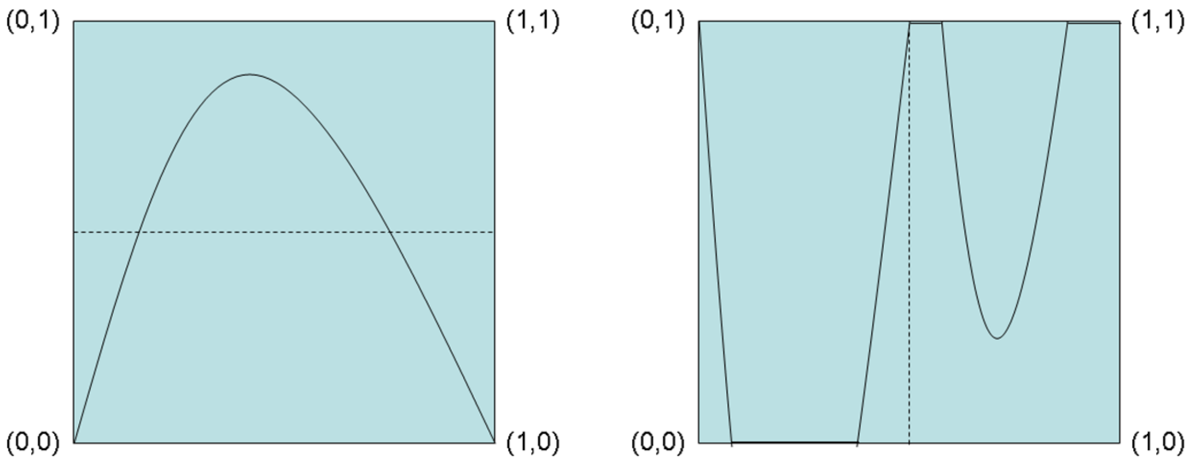

Figure 6: The figure to the left represents a function, f, in C and the line y = 1/2. The figure to the right represents $\mathbb{S}f$ and the line x = 1/2.

3.2 PRELIMINARIES

We will continue using the definitions and notation presented in chapter 2 with the exception of σ_n. Recall Alspach's mapping,

$$(Tf)(x) := \text{cut}\,(0, 1, 2f\,(2x))\,\chi_{E_{(0,1)}}(x) + \text{cut}\,(1, 2, 2f\,(2x-1))\,\chi_{E_{(1,1)}}(x).$$

Now, we define $\mathbb{S} : C \to C$ by

$$\mathbb{S}(f) := \chi_{[0,1]} - T(f),$$

for all $f \in C$ [12].

The cut function will be useful in this chapter. We will use the properties of the cut function discussed in chapter 2 and additional properties discussed here. Let $a, b, c, M \in \mathbb{R}$ where $0 \leq a < b \leq M$. Then

$$b - a - \text{cut}(a, b, c) = \text{cut}(M - b, M - a, M - c).$$

The interested reader can verify this property by considering the three cases: $c \leq a$, $c \in (a, b)$, and $c \geq b$. Furthermore, when

- a, b, c, and M are as above
- $f, g \in C$ have disjoint support, and
- $\operatorname{supp}(f) \cup \operatorname{supp}(g) = I \subset [0, 1]$,

we have the following:

$$\begin{aligned}
(b-a)\chi_I - \operatorname{cut}(a, b, f+g) &= \operatorname{cut}(M-b, M-a, M\chi_I - f - g) \\
&= \operatorname{cut}(M-b, M-a, M\chi_{supp(f)} + M\chi_{supp(g)} - f - g) \\
&= \operatorname{cut}(M-b, M-a, M\chi_{supp(f)} - f) \\
&\qquad + \operatorname{cut}(M-b, M-a, M\chi_{supp(g)} - g) \\
&= \operatorname{cut}(M-b, M-a, M\chi_{[0,1]} - f)\chi_{supp(f)} \\
&\qquad + \operatorname{cut}(M-b, M-a, M\chi_{[0,1]} - g)\chi_{supp(g)}.
\end{aligned}$$

3.3 ITERATES AND MINIMAL INVARIANT SET

To explore the powers of $\mathbb{S}$, we will first need an auxiliary function. For fixed $n \in \mathbb{N}$, take $i \in \mathbb{N}$, such that $0 \leq i < 2^{2n}$. To define σ, first write

$$i = \sum_{j=0}^{2n-1} d_j 2^j$$

with $d_j \in \{0, 1\}\ \forall j$, which is a base 2 representation. Then let

$$\sigma_{2n}(i) := \sum_{j=0}^{n-1} d_{2n-2j-2}(i) 2^{2j+1} + \sum_{j=0}^{n-1} \left(1 - d_{2n-2j-1}(i)\right) 2^{2j}.$$

So, $\sigma_2(0) = 1$, $\sigma_2(1) = 3$, $\sigma_2(2) = 0$, and $\sigma_2(3) = 2$,.

In order to use induction later, we will need a relationship between σ_{2n} and $\sigma_{2(n+m)}$. Since σ is not defined for odd subscripts, we could remove the 2. However, it is convenient to have the subscript represent the number of digits used in the binary expansions of numbers in the domain. Also, we will see that σ_{2n} is used in the formula for $\mathbb{S}^{2n}$. The needed relationship between σ_{2n}, σ_{2m}, and $\sigma_{2(n+m)}$ is given by the following lemma.

Lemma 8. *For $n, m \in \mathbb{N}$, take $j \in \mathbb{N}$, such that $0 \leq j < 2^{2m}$ and for $k \in N$ with $0 \leq k < 2^{2n}$ we have*

$$\sigma_{2n+2m}\left(2^{2m}k + j\right) = 2^{2n}\sigma_{2m}(j) + \sigma_{2n}(k).$$

Proof. Before we get too far into the proof, we should note that $d_{2m+l}\left(2^{2m}k + j\right) = d_l(k)$, for all $0 \leq l < 2n$ and $d_h\left(2^{2m}k + j\right) = d_h(j)$, for all $0 \leq h < 2m$.

Now, consider

$$\begin{aligned}
&\sigma_{2m+2n}\left(2^{2m}k + j\right) \\
&\quad = \sum_{p=0}^{m+n-1} d_{2(m+n)-2p-2}\left(2^{2m}k + j\right)2^{2p+1} + \sum_{p=0}^{m+n-1}\left(1 - d_{2(m+n)-2p-1}\left(2^{2m}k + j\right)\right)2^{2p} \\
&\quad = \sum_{p=0}^{n-1} d_{2m+2n-2p-2}\left(2^{2m}k + j\right)2^{2p+1} + \sum_{p=0}^{n-1}\left(1 - d_{2m+2n-2p-1}\left(2^{2m}k + j\right)\right)2^{2p} \\
&\qquad + \sum_{p=n}^{m+n-1} d_{2m+2n-2p-2}\left(2^{2m}k + j\right)2^{2p+1} + \sum_{p=n}^{m+n-1}\left(1 - d_{2m+2n-2p-1}\left(2^{2m}k + j\right)\right)2^{2p} \\
&\quad = \sum_{p=0}^{n-1} d_{2n-2p-2}(k)\,2^{2p+1} + \sum_{p=0}^{n-1}\left(1 - d_{2n-2p-1}(k)\right)2^{2p} \\
&\qquad + \sum_{p=n}^{m+n-1} d_{2m+2n-2p-2}(j)\,2^{2p+1} + \sum_{p=n}^{m+n-1}\left(1 - d_{2m+2n-2p-1}(j)\right)2^{2p} \\
&\quad = \sigma_{2n}(k) + \sum_{p=0}^{m-1} d_{2m-2p-2}(j)\,2^{2n+2p+1} + \sum_{p=0}^{m-1}\left(1 - d_{2m-2p-1}(j)\right)2^{2n+2p} \\
&\quad = \sigma_{2n}(k) + 2^{2n}\sigma_{2m}(j)
\end{aligned}$$

which concludes the proof of Lemma 8. □

Now, we are well prepared to prove the following:

Theorem 9. *For $n \in \mathbb{N}$ and $f \in C$,*

$$\mathbb{S}^{2n}f(x) = \sum_{i=0}^{2^{2n}-1} cut\left(\sigma_{2n}(i), \sigma_{2n}(i) + 1, 2^{2n}f\left(2^{2n}x - i\right)\right)\chi_{E_{(i,2n)}}(x).$$

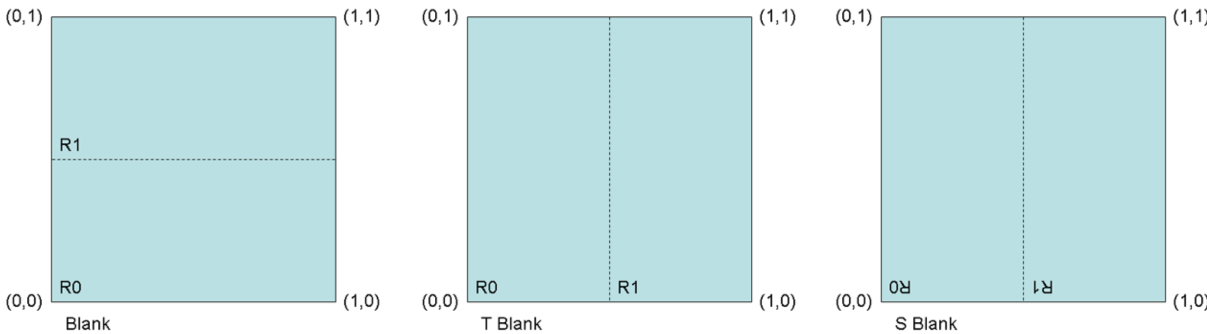

Figure 7: If the graph of a function, f, in C is broken into 2 regions as in the figure to the left, the graph of Tf can be constructed by resizing the portion of the graph of f and translating it to the appropriately labeled position in the figure to the right and adding line segments to ensure that Tf is also in C(center). Flipping the graph about the y = 1/2 axis gives $\mathbb{S}f$(right). This is denoted by the upside down region labels. Also, $\mathbb{S}$ is denoted by S in the figure.

Proof. We will use induction. To begin, we show the theorem holds for $n = 1$. Recall that

$$Tf = \sum_{i=0}^{1} \mathrm{cut}\left(i, i+1, 2f\left(2\mathrm{x} - i\right)\right) \chi_{E_{(i,1)}}.$$

So,

$$\begin{aligned}
\mathbb{S}f &= \chi_{[0,1]} - Tf \\
&= \chi_{[0,1]} - \sum_{i=0}^{1} \mathrm{cut}\left(i, i+1, 2f\left(2\mathrm{x} - i\right)\right) \chi_{E_{(i,1)}} \\
&= \sum_{i=0}^{1} \mathrm{cut}\left(1-i, 2-i, 2\chi_{[0,1]} - 2f\left(2\mathrm{x} - i\right)\right) \chi_{E_{(i,1)}}
\end{aligned}$$

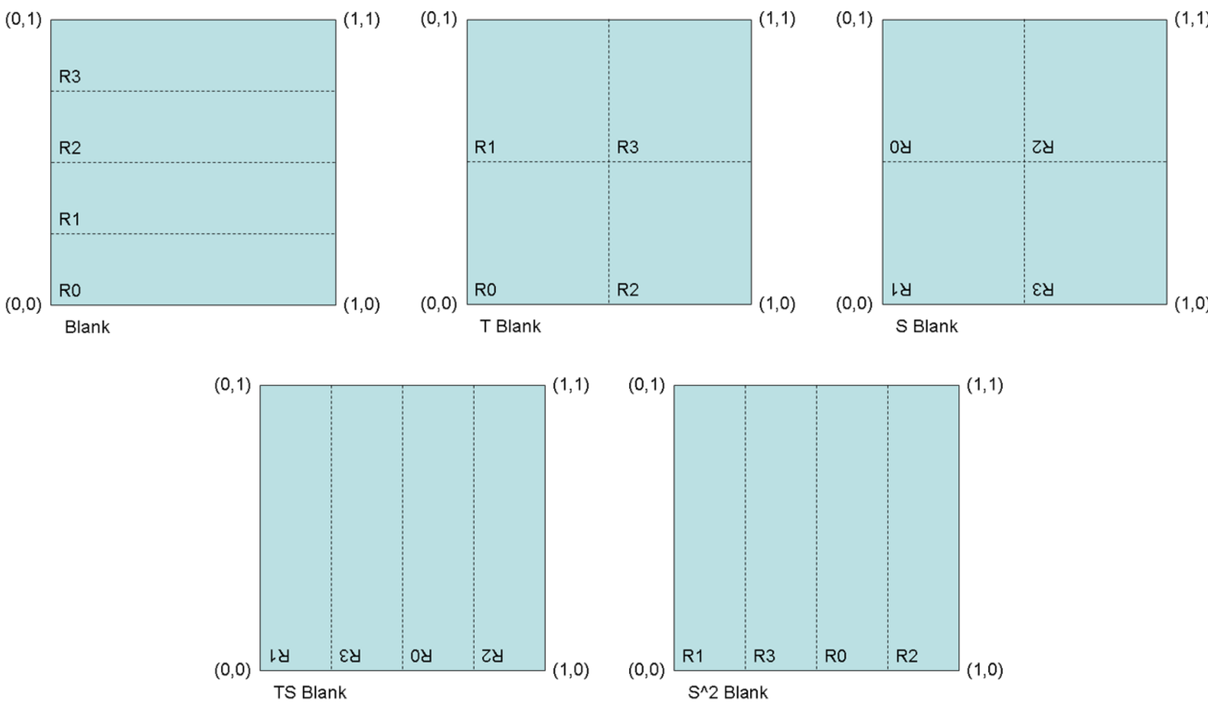

Figure 8: This figure is best interpreted in light of figures 2.3, 2.3, and 3.3. Compare the positioning of the regions with Theorem 9.

and

$$
\begin{aligned}
\mathbb{S}^2 f &= \sum_{j=0}^{1} \text{cut}\left(1-j, 2-j, 2\chi_{[0,1]} - 2\mathbb{S}f\,(2\text{x}-j)\right)\chi_{E_{(j,1)}} \\
&= \sum_{j=0}^{1} \text{cut}\Bigg(1-j, 2-j, \\
&\qquad 2\chi_{[0,1]} - 2\left[\chi_{[0,1]} - \sum_{i=0}^{1} \text{cut}\,(i, i+1, 2f\,(2\,(2\text{x}-j)-i))\,\chi_{E_{(i,1)}}\,(2\text{x}-j)\right]\Bigg)\chi_{E_{(j,1)}} \\
&= \sum_{j=0}^{1} \text{cut}\left(1-j, 2-j, 2\sum_{i=0}^{1} \text{cut}\,(i, i+1, 2f\,(2\,(2\text{x}-j)-i))\,\chi_{E_{(2j+i,2)}}\right)\chi_{E_{(j,1)}} \\
&= \sum_{j=0}^{1} \text{cut}\left(1-j, 2-j, \sum_{i=0}^{1} \text{cut}\,(2i, 2i+2, 4f\,(2\,(2\text{x}-j)-i))\,\chi_{E_{(2j+i,2)}}\right)\chi_{E_{(j,1)}} \\
&= \sum_{j=0}^{1}\sum_{i=0}^{1} \text{cut}\,(2i+1-j, 2i+2-j, 4f\,(4\text{x}-2j-i))\,\chi_{E_{(2j+i,2)}}\chi_{E_{(j,1)}} \\
&= \sum_{k=0}^{3} \text{cut}\,(\sigma_2(k), \sigma_2(k)+1, 4f\,(4\text{x}-k))\,\chi_{E_{(k,2)}},
\end{aligned}
$$

by letting $k = 2j + i$.

Now, we will assume that

$$
\mathbb{S}^{2n} f = \sum_{i=0}^{2^{2n}-1} \text{cut}\left(\sigma_{2n}(i), \sigma_{2n}(i)+1, 2^{2n} f\left(2^{2n}\text{x}-i\right)\right)\chi_{E_{(i,2n)}}
$$

holds for fixed n, and show that it holds for $n+1$. So,

$$
\begin{aligned}
\mathbb{S}^{2(n+1)} f &= \mathbb{S}^2 \mathbb{S}^{2n} f \\
&= \sum_{j=0}^{3} \operatorname{cut}\left(\sigma_2(j), \sigma_2(j)+1, 4\mathbb{S}^{2n} f\left(4\mathrm{x}-j\right)\right) \chi_{E_{(j,2)}} \\
&= \sum_{j=0}^{3} \operatorname{cut}\Bigg(\sigma_2(j), \sigma_2(j)+1, \\
&\qquad 4 \sum_{i=0}^{2^{2n}-1} \operatorname{cut}\left(\sigma_{2n}(i), \sigma_{2n}(i)+1, 2^{2n} f\left(2^{2n}\left(4\mathrm{x}-j\right)-i\right)\right) \chi_{E_{(i,2n)}}\left(4\mathrm{x}-j\right)\Bigg) \chi_{E_{(j,2)}} \\
&= \sum_{j=0}^{3} \operatorname{cut}\Bigg(\sigma_2(j), \sigma_2(j)+1, \\
&\qquad \sum_{i=0}^{2^{2n}-1} \operatorname{cut}\left(4\sigma_{2n}(i), 4\sigma_{2n}(i)+4, 2^{2n+2} f\left(2^{2n+2}\mathrm{x}-2^{2n}j-i\right)\right) \chi_{E_{(2^{2n}j+i,2n+2)}}\Bigg) \chi_{E_{(j,2)}} \\
&= \sum_{j=0}^{3} \sum_{i=0}^{2^{2n}-1} \operatorname{cut}\Big(\sigma_2(j), \sigma_2(j)+1, \\
&\qquad \operatorname{cut}\left(4\sigma_{2n}(i), 4\sigma_{2n}(i)+4, 2^{2n+2} f\left(2^{2n+2}\mathrm{x}-2^{2n}j-i\right)\right) \chi_{E_{(2^{2n}j+i,2n+2)}}\Big) \chi_{E_{(j,2)}} \\
&= \sum_{j=0}^{3} \sum_{i=0}^{2^{2n}-1} \operatorname{cut}\Big(4\sigma_{2n}(i)+\sigma_2(j), 4\sigma_{2n}(i)+\sigma_2(j)+1, \\
&\qquad 2^{2n+2} f\left(2^{2n+2}\mathrm{x}-2^{2n}j-i\right)\Big) \chi_{E_{(2^{2n}j+i,2n+2)}} \\
&= \sum_{j=0}^{3} \sum_{i=0}^{2^{2n}-1} cut\Big(\sigma_{2n+2}(2^{2n}j+i), \sigma_{2n+2}(2^{2n}j+i)+1, \\
&\qquad 2^{2n+2} f\left(2^{2n+2}\mathrm{x}-2^{2n}j-i\right)\Big) \chi_{E_{(2^{2n}j+i,2n+2)}} \\
&= \sum_{k=0}^{2^{2n+2}-1} cut\left(\sigma_{2n+2}(k), \sigma_{2n+2}(k)+1, 2^{2n+2} f\left(2^{2n+2}\mathrm{x}-k\right)\right) \chi_{E_{(k,2n+2)}},
\end{aligned}
$$

where $k = 2^{2n}j + i$. This concludes the induction and our proof. □

Lemma 10. *For any $f \in C$ and $s \in S$,*

$$\lim_{m\to\infty} \int_{[0,1]} \mathbb{S}^{2m} f \cdot s = \|f\|_1 \int_{[0,1]} s \ .$$

Proof. Since $s \in S$ is a finite sum of constant functions on intervals of the form $E_{(i,n)}$, it suffices to show Lemma 10 holds for $s = \chi_{E_{(l,2n)}}$, where $n \in \mathbb{N}$ and $0 \leq l < 2^{2n}$. Fix $m \in \mathbb{N}$. We have that

$$\begin{aligned}
&\int_{[0,1]} \mathbb{S}^{2(n+m)} f \chi_{E_{(l,n)}} \\
&\quad = \int_{[0,1]} \Bigg(\sum_{i=0}^{2^{2n+2m}-1} \mathrm{cut}\big(\sigma_{2n+2m}(i), \sigma_{2n+2m}(i)+1, \\
&\qquad\qquad 2^{2n+2m} f\left(2^{2n+2m}\mathrm{x} - i\right)\big)\chi_{E_{(i,2n+2m)}} \Bigg) \chi_{E_{(l,2n)}} \\
&\quad = \sum_{i=2^{2m}l}^{2^{2m}l+2^{2m}-1} \int_{[0,1]} \mathrm{cut}\left(\sigma_{2n+2m}(i), \sigma_{2n+2m}(i)+1, 2^{2n+2m} f\left(2^{2n+2m}\mathrm{x} - i\right)\right)\chi_{E_{(i,2n+2m)}} \\
&\quad = \frac{1}{2^{2n+2m}} \sum_{i=2^{2m}l}^{2^{2m}l+2^{2m}-1} \int_{[0,1]} \mathrm{cut}\left(\sigma_{2n+2m}(i), \sigma_{2n+2m}(i)+1, 2^{2n+2m} f(\mathrm{x})\right).
\end{aligned}$$

From here we wish to reorder the terms in the summation. To that end, define

$$B := \left\{ j \in \mathbb{N} | 0 \leq j < 2^{2m} \right\}$$

and

$$A := \left\{ 2^{2m} l \right\} + B.$$

Notice that we are summing over A. Now, using Lemma 8

$$\sigma_{2m+2n}(A) = \sigma_{2m+2n}(\left\{2^{2m}l\right\} + B) = 2^{2n}\sigma_{2m}(B) + \left\{\sigma_{2n}(l)\right\}.$$

So,

$$\begin{aligned}
&\int_{[0,1]} \mathbb{S}^{2(n+m)} f \chi_{E_{(l,n)}} \\
&\quad = \frac{1}{2^{2n+2m}} \sum_{j=0}^{2^{2m}-1} \int_{[0,1]} \mathrm{cut}\left(2^{2n}\sigma_{2m}(j) + \sigma_{2n}(l), 2^{2n}\sigma_{2m}(j) + \sigma_{2n}(l) + 1, 2^{2n+2m} f\right) \\
&\quad = \frac{1}{2^{2n+2m}} \sum_{j=0}^{2^{2m}-1} \int_{[0,1]} \mathrm{cut}\left(2^{2n} j + \sigma_{2n}(l), 2^{2n} j + \sigma_{2n}(l) + 1, 2^{2n+2m} f\right),
\end{aligned}$$

where the last equality is the reordering.

By the same argument used in Lemma 2, we see that

$$\left|\int_{[0,1]} \mathbb{S}^{2(n+m)} f \chi_{E_{(l_1,2n)}} - \int_{[0,1]} \mathbb{S}^{2(n+m)} f \chi_{E_{(l_2,2n)}}\right| \leq \frac{1}{2^{2n+2m}}$$

for $l_1, l_2 \in \mathbb{N}$ with $0 \leq l_1 < 2^{2n}$ and $0 \leq l_2 < 2^{2n}$. Also, it is easy to verify that

$$\int_{[0,1]} f = \int_{[0,1]} \mathbb{S}^{2n+2m} f = \sum_{k=0}^{2^{2n}-1} \int_{[0,1]} \mathbb{S}^{2n+2m} f \chi_{E_{(k,2n)}}.$$

Now,

$$\begin{aligned}
&\left|\int_{[0,1]} \mathbb{S}^{2m+2n} f \chi_{E_{(l,2n)}} - \frac{1}{2^{2n}} \int_{[0,1]} f\right| \\
&\quad = \frac{1}{2^{2n}} \left| 2^{2n} \int_{[0,1]} \mathbb{S}^{2m+2n} f \chi_{E_{(l,2n)}} - \sum_{k=0}^{2^{2n}-1} \int_{[0,1]} \mathbb{S}^{2m+2n} f \chi_{E_{(k,2n)}} \right| \\
&\quad \leq \frac{1}{2^{2n}} \sum_{k=0}^{2^{n}-1} \left| \int_{[0,1]} \mathbb{S}^{2m+2n} f \chi_{E_{(l,2n)}} - \int_{[0,1]} \mathbb{S}^{2m+2n} f \chi_{E_{(k,2n)}} \right| \\
&\quad \leq \frac{1}{2^{2n}} 2^{2n} \frac{1}{2^{2n+2m}} = \frac{1}{2^{2n+2m}} \to 0\,, \text{ as } m \to \infty\,;
\end{aligned}$$

which concludes Lemma 10.

□

Theorem 11. *For all $f \in C$, $\mathbb{S}^{2n} f$ converges weakly to $\|f\|_1 \chi_{[0,1]}$ and $\mathbb{S}^{2n+1} f$ converges weakly to $(1 - \|f\|_1) \chi_{[0,1]}$.*

After noticing that $\|\mathbb{S} f\|_1 = 1 - \|f\|_1$, this theorem follows directly from Lemma 10 using the same argument as Theorem 3.

Lemma 12. *For every $f \in C$, $\frac{1}{2}\chi_{[0,1]} \in D_\infty(f)$ and $D_\infty\left(\frac{1}{2}\chi_{[0,1]}\right) \subseteq D_\infty(f)$.*

Proof. Take $f \in C$. From Theorem 11, $\mathbb{S}^{2n} f$ converges weakly to $\|f\|_1 \chi_{[0,1]}$ and $\mathbb{S}^{2n+1} f$ converges weakly to $(1 - \|f\|_1) \chi_{[0,1]}$. So,

$$\|f\|_1 \chi_{[0,1]} \in \overline{conv} \left(\cup_{n \in \mathbb{N}} \{\mathbb{S}^{2n} f\} \right) \subseteq D_\infty(f)$$

and

$$(1 - \|f\|_1) \chi_{[0,1]} \in \overline{conv} \left(\cup_{n \in \mathbb{N}} \{\mathbb{S}^{2n+1} f\} \right) \subseteq D_\infty(f).$$

Clearly,

$$\frac{1}{2} \chi_{[0,1]} = \frac{1}{2} \|f\|_1 \chi_{[0,1]} + \frac{1}{2} (1 - \|f\|_1) \chi_{[0,1]} \subseteq D_\infty(f),$$

because $D_\infty(f)$ is convex. $D_\infty(f)$ is also $\mathbb{S}$-invariant and closed. Thus, $D_n(\frac{1}{2}\chi_{[0,1]}) \subseteq D_\infty(f)$ for every $n \in \mathbb{N}$ and $D_\infty(\frac{1}{2}\chi_{[0,1]}) \subseteq D_\infty(f)$. □

Theorem 13. *$D_\infty\left(\frac{1}{2}\chi_{[0,1]}\right)$ is the unique minimal invariant subset of $(\mathbb{S}, C)$.*

Proof. Obviously, $D_\infty(\frac{1}{2}\chi_{[0,1]})$ is non-empty, closed, convex, and $\mathbb{S}$-invariant. Suppose M is a non-empty, closed, convex, invariant subset of $D_\infty(\frac{1}{2}\chi_{[0,1]})$. Choose any $f \in M$. Recall that $D_\infty(f) \subset M$. Lemma 12 implies $D_\infty\left(\frac{1}{2}\chi_{[0,1]}\right) \subseteq M$. So, $M = D_\infty\left(\frac{1}{2}\chi_{[0,1]}\right)$. Therefore, $D_\infty\left(\frac{1}{2}\chi_{[0,1]}\right)$ is a minimal invariant set. Let $B \subseteq C$ be any minimal invariant set. There is an $f \in B$, because B is non-empty. So, $D_\infty(\frac{1}{2}\chi_{[0,1]}) \subseteq D_\infty(f) = B$. Thus, $D_\infty(\frac{1}{2}\chi_{[0,1]})$ is the unique minimal invariant set for $(\mathbb{S}, C)$. □

Theorem 14. *Sine's mapping, $\mathbb{S}$, is fixed point free on C.*

Proof. First, recall that the singleton containing any fixed point must be minimal invariant. Now, assume that $\mathbb{S}$ has a fixed point in C. Since $D_\infty\left(\frac{1}{2}\chi_{[0,1]}\right)$ is the only minimal invariant subset of C by Theorem 13, it must be the singleton containing the fixed point. However, $\mathbb{S}(\frac{1}{2}\chi_{[0,1]}) = \chi_{[\frac{1}{2},1]} \neq \frac{1}{2}\chi_{[0,1]}$, which give the contradiction. Thus, $\mathbb{S}$ is fixed point free on C. □

3.4 DISCUSSION OF SINE'S MAPPING

Here, as in Chapter 2, we do not use Zorn's lemma or any equivalent statement. This is possible because we have a formula for the iterates of $\mathbb{S}^2$. The formula actually leads to much more that just the removal of a set theoretic axiom.

Without a formula for the iterates of $\mathbb{S}$, it is relatively easy to see that all minimal invariant sets of $\mathbb{S}$ must be subsets of $C_{\frac{1}{2}}$. However, the number, geometry, and elements of such sets were hard to even guess. Now, the minimal invariant set, $D_\infty\left(\frac{1}{2}\chi_{[0,1]}\right)$, can be built from below using the definite of D_∞. Moreover, any invariant superset of $D_\infty\left(\frac{1}{2}\chi_{[0,1]}\right)$ can be used to exclude some elements of C from belonging to the minimal invariant set as well.

There are similarities between (T, C) and $(\mathbb{S}, C)$, and a few important differences. $\mathbb{S}$ is actually fixed point free on all of C, whereas T is not. This makes Sine's mapping somewhat more functionally useful. Also, T has a family of minimal invariant sets, whereas $\mathbb{S}$ has a unique minimal invariant set. This makes T a perfect example to have in mind while reading [6], since it explores characteristics of parallel families of minimal invariant sets.

4.0 KOMLÓS SETS IN BANACH FUNCTION SPACES

4.1 INTRODUCTION

In the Scottish handbook, H. Steinhaus asks if there exists a family, F, of measurable functions defined on a measure space (X, μ) such that $|f(x)| = 1$ for all $x \in X$ and $f \in F$ and for each sequence $\{f_n\} \in F$ the sequence of averages,

$$\frac{1}{n}\sum_{k=1}^{n} f_k(x),$$

is divergent for almost all x. D. G. Austin showed that when restricted to zero-one valued functions, the answer is no. A. Rényi answered the question with no restrictions. Then, Révész showed that it was enough to assume $M(f^2) \leq K$ for $f \in F$ instead of $|f(t)| = 1$, where $M(f^2) \equiv \|f\|_2 \equiv \int_X |f(x)|^2 d\mu(x)$. See [9]. Finally, Komlós [9] proved the following:

Theorem 15. *For any sequence $\{f_n\}_n \subset L_1(\mu)$, μ a probability measure, with $\|f_n\| \leq M < \infty$, there exists a subsequence $\{g_n\}_n$ of $\{f_n\}_n$ and a $g \in L_1(\mu)$ such that for any further subsequence $\{h_n\}_n$ of $\{g_n\}_n$*

$$\frac{1}{n}\sum_{i=1}^{n} h_i \to g \ a.e.$$

We will say that a sequence $\{f_n\}_n \in L_1$ is *Komlós* if there exists a subsequence $\{g_n\}_n$ of $\{f_n\}_n$ and a $g \in L_1$ such that for any further subsequence $\{h_n\}_n$ of $\{g_n\}_n$

$$\frac{1}{n}\sum_{i=1}^{n} h_n \to g \ a.e.$$

Theorem 15 can now be rephrased to read: For μ a probability measure, every norm-bounded sequence in $L_1(\mu)$ is Komlós. Later, we will generalize this theorem to σ-finite measures and a broad class of Banach spaces.

So, this leads us to the discussion of a converse. Lennard [10] provided an example of a sequence that is Komlós and not norm-bounded in $L_1(\mu)$. Observe $f_n = n^2\chi_{[0,\frac{1}{n}]}$. This, together with the history of the problem might prompt one to generalize the notion of Komlós from sequences to sets. We will say that the set C is *Komlós* if every sequence $\{f_n\}_n \subset C$ is Komlós and the corresponding g is in C as well. Lennard showed that every convex Komlós subset of $L_1(\mu)$ is norm-bounded [10].

4.2 PRELIMINARIES

To continue further, we will develop a definition of Banach function norms. Let (Ω, μ) be a measure space. Denote the μ-measurable functions on Ω taking values in $\mathbb{R}$ by $\mathcal{M}$ and the cone of μ-measurable functions on Ω taking values in $[0,\infty]$ by $\mathcal{M}^+$. Also, let χ_E represent the characteristic function on E, a μ-measurable function. Now, a mapping $\rho : \mathcal{M}^+ \to [0,\infty]$ is called a function norm if for all $f, g, \{f_n\}_{n\in\mathbb{N}}$ in $\mathcal{M}^+$, for every $\alpha \geq 0$, and for E μ-measurable the following hold:

- (P1) $\rho(f) = 0 \Leftrightarrow f = 0$, $\rho(\alpha f) = \alpha\rho(f)$, and $\rho(f+g) \leq \rho(f) + \rho(g)$
- (P2) $0 \leq g \leq f \Rightarrow \rho(g) \leq \rho(f)$
- (P3) $\mu(E) < \infty \Rightarrow \rho(\chi_E) < \infty$.

So, for ρ, a function norm, the collection X of all functions, f, in $\mathcal{M}$ such that $\rho(|f|) < \infty$ is called a *Köthe* space, where $\|f\|_X \equiv \rho(|f|)$ for all $f \in X$. A complete *Köthe* space is a *Banach function space*. We will omit subscripts when there should be no confusion.

It will be useful to discuss Banach function spaces with nice properties. For example, we will say that a Banach function space satisfies the *Fatou property* if for all $f, f_n \in \mathcal{M}$,

$$0 \leq f_n \uparrow f \ \mu - a.e. \Rightarrow \rho(f_n) \uparrow \rho(f).$$

Also, a Banach function space satisfying

$$\mu(E) < \infty \Rightarrow \int_E |f| \leq C_E\rho(|f|), \text{ for all } f \in X, \text{ for some } 0 < C_E < \infty$$

will be called a FI Banach function space. Note that $L_p(\mathbb{R})$ with the Lebesgue measure is an FI (finitely integrable) Banach function space with the Fatou property for all $1 \leq p \leq \infty$. For further development of the concept of Banach function spaces we refer the reader to [2, 13].

Let (Ω, μ) be a measure space and $(X, \|\cdot\|)$ be a Banach function space with underlying measure space Ω. Recall that we say a set $S \subset X$ is Komlós if for every sequence $\{f_n\}_{n=1}^{\infty}$ in S, there exists a subsequence $\{g_n\}_{n=1}^{\infty}$ of $\{f_n\}_{n=1}^{\infty}$ and a function $g \in S$ such that for any further subsequence $\{h_n\}_{n=1}^{\infty}$ of $\{g_n\}_{n=1}^{\infty}$

$$\frac{1}{N}\sum_{m=1}^{N} h_m \underset{N}{\rightarrow} g \ a.e.$$

4.3 RESULTS

We will begin by generalizing Theorem 15. First, one should note that Theorem 15 can be generalized from μ a probability space to μ a finite measure space by a simple change of measure.

Theorem 16. *Let X be an FI Banach function space with the Fatou property defined on (Ω, μ), a σ-finite measure space. For any sequence $\{f_n\}_n \subset X$ with $\|f_n\| \leq M < \infty$, there exists a subsequence $\{g_n\}_n$ of $\{f_n\}_n$ and a $g \in X$ with $\|g\| \leq M$ such that for any further subsequence $\{h_n\}_n$ of $\{g_n\}_n$*

$$\frac{1}{n}\sum_{i=1}^{n} h_i \rightarrow g \ \mu - a.e.$$

Proof. Fix $\{f_n\}_n$. Let $\Omega = \cup_{n\in\mathbb{N}}\{\Omega_n\}$ where Ω_n has finite measure. Define the Banach function spaces $X_n \equiv \{f|_{\Omega_n} : f \in X\}$ with underlying measure μ and the inherited Banach function norm. Because X is an FI Banach function space, we have that $\int_{\Omega_j} |f_n| \leq C_j\rho(|f_n|) \leq C_j M$ for some fixed $0 < C_j < \infty$ for all n. This allows us to employ Theorem 15. First, we use Theorem 15 to obtain a subsequence $\{g_{(1,n)}\}_n$ of $\{f_n\}_n$ and a $g_1 \in X_1$ such that for any subsequence $\{h_n\}_n$ of $\{g_{(1,n)}\}_n$ we have

$$\frac{1}{n}\sum_{i=1}^{n} h_i \rightarrow g_1 \ a.e. \text{ in } \Omega_1.$$

Then, we can proceed inductively. Using Theorem 15 again gives us, $\{g_{(j+1,n)}\}_n$, a subsequence of $\{g_{(j,n)}\}_n$ and a $g_{j+1} \in X_{j+1}$ such that for any subsequence $\{h_n\}_n$ of $\{g_{(j+1,n)}\}_n$ we have

$$\frac{1}{n}\sum_{i=1}^{n} h_i \to g_{j+1} \ a.e. \text{ in } \Omega_{j+1}.$$

Without loss of generality, we can impose the restriction that $g_{(j,n)} = g_{(j+1,n)}$ for $n \leq j+1$. It is easy to verify that g, defined by $g(x) \equiv g_j(x)$ when $x \in \Omega_j$, is in $\mathcal{M}$ for (Ω, μ). Moreover, $\{g_{(n,n)}\}_n$ is a subsequence of $\{g_{(j,n)}\}_n$ for all j. So, for any subsequence $\{h_n\}_n$ of $\{g_{(n,n)}\}_n$ we have

$$\frac{1}{n}\sum_{i=1}^{n} h_i \to g \ a.e. \text{ in } \Omega.$$

All that is left is to show that $g \in X$. To this end, we will define k_n by

$$k_n(x) \equiv \inf\left(\{|\frac{1}{j}\sum_{i=1}^{j} g_{(i,i)}(x)| : j \geq n\} \cup \{|g(x)|\}\right).$$

Clearly, $k_n \in \mathcal{M}$ and $\|k_n\| \leq \left\|\frac{1}{n}\sum_{i=1}^{n} g_{(i,i)}\right\| \leq M$. Since $k_n \uparrow g \ \mu - a.e.$, $\rho(|k_n|) \uparrow \rho(|g|)$. Therefore $\|g\| \leq M$.

$\square$

To see the importance of the Fatou property in this theorem, consider the following example. Let $\Omega = \mathbb{N}$ with $\mu(S) = |S|$, where $|S|$ denotes the order of S. We will use the standard notation: $x_n \equiv x(n)$. Now, define

$$\rho(x) \equiv \begin{cases} \infty & \text{not } [\lim_n x_n = 0] \\ \sup_n |x_n| & \lim_n x_n = 0. \end{cases}$$

It is easy to see that $X = c_0$ with the usual norm. So, if we define the sequence $f_n \in X$ by $f_n(j) \equiv 1$ if $j \leq n$ and zero otherwise, then $f_n(j) \uparrow 1$ for all j and $\rho(\chi_{\mathbb{N}}) = \infty$. It can be verified that this is an example of a norm bounded sequence in a Banach function space (without the Fatou property) that fails to be Komlós; i.e., it fails the conclusion of Theorem 16.

From here we begin the presentation of a converse for Theorem 16.

Theorem 17. *Let X be a Banach function space satisfying the Fatou property with a finite underlying measure space (Ω, μ). Suppose C is a convex Komlós subset of X. Then C is norm bounded.*

Proof. To obtain a contradiction, suppose that C is not norm bounded. Then, there exists a sequence $\{g_n\}_{n=1}^{\infty}$ in C such that $\|g_n\| \to \infty$.

C is Komlós. So, by passing to a subsequence if necessary, we have $g \in C$ such that

$$\frac{1}{N}\sum_{m=1}^{N} h_m \underset{N}{\rightarrow} g \text{ a.e.}$$

for every subsequence $\{h_m\}_m$ of $\{g_n\}_n$.

We can shift $\{g_n\}_n$ for convenience. C convex $\Rightarrow C - g$ is convex. Also $C \subset X \Rightarrow C - g \subset X$, and C norm bounded $\Rightarrow C - g$ is norm bounded. It is worth noting that $g \in C$ implies that $\theta \in C - g$, where θ denotes the zero function. Lastly, we have

$$\frac{1}{N}\sum_{m=1}^{N} (h_m - g) \underset{N}{\rightarrow} \theta,$$

for every subsequence $\{h_m\}_m$ of $\{g_n\}_n$.

To finish our shift, let us relabel $g_n - g$ as g_n and $C - g$ as C. Now, we have $\|g_n\| \to \infty$, $\theta \in C$, and for every subsequence $\{h_m\}_m$ of $\{g_n\}_n$,

$$\frac{1}{N}\sum_{m=1}^{N} h_m \underset{N}{\rightarrow} \theta, a.e.. \tag{4.1}$$

Now, we will construct a sequence $\{f_n\}_n$ from $\{g_n\}_n$. First, let $u_1 := 1$ and $f_1 := g_{u_1}$. Since $\|g_n\| \to \infty$, there exists a $u_2 \in \mathbb{N}$ such that $u_2 > u_1$ and

$$\|g_{u_2}\| > \|g_{u_1}\| + 2(2^2).$$

Define f_2 by

$$f_2 := \frac{1}{2}(g_{u_1} + g_{u_2}).$$

Note that $f_2 \in C$ and

$$\|f_2\| \geq \frac{1}{2}(\|g_{u_2}\| - \|g_{u_1}\|) > \frac{1}{2} \cdot 2(2^2).$$

In general, define

$$f_n := \frac{1}{n}\sum_{j=1}^{n} g_{u_j},$$

where $u_n \in \mathbb{N}$ is chosen such that $u_n > u_{n-1}$ and

$$\|g_{u_n}\| > \left\|\sum_{j=1}^{n-1}(g_{u_j})\right\| + n(2^n).$$

We see that

$$\|f_n\| = \left\|\frac{1}{n}\sum_{j=1}^{n}(g_{u_j})\right\| \geq \frac{1}{n}\left(\|g_{u_n}\| - \left\|\sum_{j=1}^{n-1}(g_{u_j})\right\|\right) \geq 2^n.$$

Also, from equation (4.1), we have $f_n \underset{n}{\rightarrow} \theta, a.e.$. We can now construct a subsequence of $\{f_n\}_n$ by inductively defining a strictly increasing sequence $\{n_k\}_{k=1}^{\infty}$ in $\mathbb{N}$, a non-increasing sequence $\{E_n\}_{n=0}^{\infty}$ of μ-measurable sets, and a non-increasing sequence $\{\delta_k\}_{k=0}^{\infty}$ of positive real numbers such that the following hold:

1. $\delta_k < \frac{\delta_{k-1}}{2}$
2. For E μ-measurable with $\mu(E) < \delta_k$, $\left\|f_{n_k}\chi_{E_k \setminus E}\right\| > 2^k\left(\sum_{j=0}^{k-1}\left\|f_{n_j}\right\| + 2\|\chi_\Omega\| + 2^k\right)$
3. $\left\|f_n \chi_{E_{k-1}\setminus E_k}\right\|_\infty < 1$, $\forall\ n \geq n_k$
4. $\mu(E_k) < \delta_{k-1}$.

To begin, let $E_0 = \Omega$, $\delta_0 = 2\mu(\Omega)$, and $n_0 = 1$. Let $E_1 = \Omega$. We have $\|f_n\| \to \infty$. So, let $n_1 \in \mathbb{N}$ with $n_1 > n_0$ such that

$$\|f_{n_1}\chi_{E_1}\| > 2^1\left(\|f_{n_0}\| + 2\|\chi_\Omega\| + 2^1\right) + 1$$

and

$$\left\|f_n \chi_{E_0\setminus E_1}\right\|_\infty < 1, \forall n \geq n_1. \tag{4.2}$$

Note that our choice of E_0 and E_1 makes restriction (4.2) trivial. By the Fatou property, there exists a $\delta_1 \in (0, \mu(\Omega))$ such that for every measurable E with $\mu(E) < \delta_1$ we have $\left\|f_{n_1}\chi_{E_1\setminus E}\right\| > 2^1\left(\|f_{n_0}\| + 2\|\chi_\Omega\| + 2^1\right)$. Obviously, $\mu(E_1) < \delta_0$. We can proceed inductively from here.

Fix $m \in \mathbb{N}$ with $m > 1$ and assume that we have constructed sequences $\{n_k\}_{k=0}^{m-1}$, $\{E_k\}_{k=0}^{m-1}$, and $\{\delta_k\}_{k=1}^{m-1}$ that satisfy their respective requirements for each $\mathbb{N} \ni k < m$. Recall, $f_n \underset{n}{\to} \theta$ almost everywhere and, in particular, on E_{m-1}. By Egoroff's theorem, there exists a measurable $E_m \subset E_{m-1}$ such that

$$\mu(E_m) < \delta_{m-1} \text{ and } \left\| f_n \chi_{E_{m-1} \setminus E_m} \right\|_\infty \underset{n}{\to} 0.$$

Also, note that property 3 holds for every $k \leq m-1$, which implies

$$\left\| f_n \chi_{\Omega \setminus E_{m-1}} \right\|_\infty < 1, \ \forall\, n \geq n_{m-1}.$$

Take $\hat{n} > n_{m-1}$ s.t.

$$\left\| f_n \chi_{E_{m-1} \setminus E_m} \right\|_\infty < 1, \ \forall\, n \geq \hat{n}.$$

For all $n \geq \hat{n} \geq n_{m-1}$,

$$\| f_n \chi_{E_m} \| \geq \| f_n \| - \left\| f_n \chi_{\Omega \setminus E_m} \right\| \geq \| f_n \| - \| \chi_\Omega \| .$$

We have $\| f_n \| \underset{n}{\to} \infty$. Thus,

$$\lim_{n \to \infty} \| f_n \chi_{E_m} \| = \infty.$$

Choose $n_m \in \mathbb{N}$ with $n_m > \hat{n} > n_{m-1}$ so that

$$\| f_{n_m} \chi_{E_m} \| > 2^m \left(\sum_{j=0}^{m-1} \left\| f_{n_j} \right\| + 2 \| \chi_\Omega \| + 2^m \right) + 1.$$

Using the Fatou property, we have the existence of a $\delta_m < \frac{\delta_{m-1}}{2}$ such that for every measurable E with $\mu(E) < \delta_m$ we have

$$\left\| f_{n_m} \chi_{E_m \setminus E} \right\| > 2^m \left(\sum_{j=0}^{m-1} \left\| f_{n_j} \right\| + 2 \| \chi_\Omega \| + 2^m \right).$$

Thus, the induction is complete.

Now, by eliminating terms in the sequence we can, without loss of generality, assume that $\delta_1 < \frac{1}{2^2}$. Thus $\mu(E_k) < \frac{1}{2^k}$ for $k \geq 2$. For notational simplicity, let us relabel f_{n_k} as f_n. We will refer to the four properties as they apply to the new f_n as 1^* through 4^* in order to remind the reader that with our new notation the subsequence $f_n \equiv f_{n_k}$.

Define

$$\psi_k \equiv \sum_{j=1}^{k} \frac{1}{2^j} f_j,$$

for every $k \in \mathbb{N}$. Recall that $\theta \in C$ and that C is convex. So, we have $\psi_k \in C$ for every $k \in \mathbb{N}$. Since C is Komlós, there exists a subsequence, ψ_{k_l}, of ψ_k and a $q \in C$ such that

$$q_n \equiv \frac{1}{n} \sum_{l=1}^{n} \psi_{k_l} \underset{n}{\rightarrow} q \ a.e.$$

and also,

$$q \in C \Rightarrow q \in X \Rightarrow \|q\| < \infty.$$

It is not difficult to show

$$q_n = \sum_{j=1}^{n} \frac{n-j+1}{n} \sum_{t=k_{j-1}+1}^{k_j} \frac{1}{2^t} f_t.$$

Henceforth, all inequalities involving functions will be true almost everywhere. Fix $m \in \mathbb{N}$, $m > 1$. There exists a unique $i \in \mathbb{N}$ such that $k_{i-1} < m \leq k_i$. For notational convenience, let $c_m := \chi_{(E_m \setminus E_{m+1})}$. Taking any $n \geq i$ we observe

$$\begin{aligned}
|q_n| c_m &= \left| \sum_{j=1}^{n} \frac{n-j+1}{n} \sum_{t=k_{j-1}+1}^{k_j} \frac{1}{2^t} f_t \right| c_m \\
&= \left| \frac{1}{2^m} \frac{n-i+1}{n} f_m + \sum_{j=1}^{n} \sum_{t=k_{j-1}+1, t \neq m}^{k_j} \frac{1}{2^t} \frac{n-j+1}{n} f_t \right| c_m \\
&\geq \left(\frac{1}{2^m} \frac{n-i+1}{n} |f_m| - \sum_{t=1, t\neq m}^{k_n} \frac{1}{2^t} |f_t| \right) c_m \\
&\geq \left(\frac{1}{2^m} \frac{n-i+1}{n} |f_m| \right) c_m - \sum_{t=1}^{m-1} \frac{1}{2^t} |f_t| c_m - \sum_{t=m+1}^{k_n} \frac{1}{2^t} |f_t| c_m.
\end{aligned}$$

Recall that by property 3*, $|f_t| < 1$, for every $t > m$ on $E_m \backslash E_{m+1}$. So, to summarize, $\forall\, m \in \mathbb{N}$, $\exists\, i \in \mathbb{N}$, s.t. for all $n \geq i$ on the set $E_m \backslash E_{m+1}$ we have

$$\begin{aligned} |q_n| &\geq \frac{1}{2^m}\frac{n-i+1}{n}|f_m| - \sum_{t=1}^{m-1}\frac{1}{2^t}|f_t| - 1 \\ &= \frac{1}{2^m}|f_m| - \sum_{t=1}^{m-1}\frac{1}{2^t}|f_t| - 1 - \frac{i-1}{n}|f_m|. \end{aligned}$$

Since this inequality holds for all large n, we can take the limit as $n \to \infty$. On $E_m \backslash E_{m+1}$, we have

$$|q| \geq \frac{1}{2^m}|f_m| - \sum_{t=1}^{m-1}\frac{1}{2^t}|f_t| - 1.$$

Let $\gamma_m \equiv \frac{1}{2^m}|f_m| - \sum_{t=1}^{m-1}\frac{1}{2^t}|f_t| - 1$ and we will use the usual notation that $f^+(x) \equiv \max\{f(x), 0\}$ and $f^-(x) \equiv \max\{-f(x), 0\}$. So, $|q|\, c_m \geq \gamma_m^+\, c_m$ and $\gamma_m^+ = |\gamma_m| - \gamma_m^-$. Note that $\gamma_m^-\, c_m \leq \left(\sum_{t=1}^{m-1}\frac{1}{2^t}|f_t| + 1\right) c_m$. Therefore, by property (P2),

$$\begin{aligned} \|q\| &\geq \|\, q\, c_m\| \\ &\geq \left\|\gamma_m^+\, c_m\right\| \\ &\geq \|\gamma_m\, c_m\| - \left\|\gamma_m^-\, c_m\right\| \\ &\geq \left\|\left(\frac{1}{2^m}|f_m| - \sum_{t=1}^{m-1}\frac{1}{2^t}|f_t| - 1\right) c_m\right\| - \left\|\left(\sum_{t=1}^{m-1}\frac{1}{2^t}|f_t| + 1\right) c_m\right\| \\ &\geq \left\|\frac{1}{2^m} f_m c_m\right\| - 2\left(\sum_{t=1}^{m-1}\frac{1}{2^t}\,\|f_t c_m\| + \|c_m\|\right) \\ &\geq \frac{1}{2^m}\,\|f_m c_m\| - \sum_{t=1}^{m-1}\|f_t\| - 2\,\|\chi_\Omega\|\,. \end{aligned}$$

Finally, using 2* we see that

$$\|q\| \geq 2^m.$$

This holds for arbitrary m. Therefore, q is not in X; however, previous work showed that q is in X. That provides the desired contradiction and concludes Theorem 17. □

It is possible to generalize Theorem 17 from μ a finite measure to μ a σ-finite measure. This is shown next.

Theorem 18. *Let X be a Banach function space satisfying the Fatou property with a σ-finite underlying measure space (Ω, μ). Suppose C is a convex Komlós subset of X. Then C is norm bounded.*

Proof. Since (Ω, μ) is σ-finite we can take $g \in L_1(\mu)$ s.t. $0 < g \leq 1$ and $\int g d\mu \leq 1$. Define $d\nu = g d\mu$, so that $\nu(S) = \int_S g d\mu$. Now, define $r(f) \equiv \rho(fg)$. It is easily verified that r is a Banach function norm. Call the emerging Banach function space Y. Then, (Y, r) is a Banach function space with underlying measure (Ω, ν) that satisfies the Fatou property. Now, we can define $T : X \to Y$ by $T(f) \equiv fg^{-1}$. Clearly, T is an isometry from X onto Y. So, $T(C)$ is a convex Komlós subset of Y. Since $\nu(\Omega) \leq 1$, we can apply Theorem 17 to obtain a norm bound, M, for $T(C)$, which is also a norm bound for C, since T is an isometry.

□

We should point out that the proof of Theorem 18 generalizes from Theorem 2.2 of [10] by a relatively simple change in notation.

4.4 CONCLUSIONS

The strength of Theorem 17 and Theorem 18 relies on the existence of convex Komlós sets. Thankfully, Theorem 16 shows that the unit ball of X (a FI Banach function space with the Fatou property defined on (Ω, μ), a σ-finite measure space) is a Komlós set. Clearly, the translation and dilation of Komlós sets are also Komlós sets. This gives us a large family of convex Komlós sets.

Chatterji [3] proved the following:

Theorem 19. *Let (Ω, μ) be a measure space and $\{f_n\}_n \in L_p(\Omega, \mu), 0 < p < 2$ be a norm bounded sequence. Then, there exists a subsequence $\{g_n\}_n$ of $\{f_n\}_n$ and a function $g \in L_p$*

such that for any subsequence $\{h_n\}_n$ *of* $\{g_n\}_n$

$$\lim_{n\to\infty} n^{-\frac{1}{p}} \sum_{k=1}^{n} (h_n - g) = 0 \ a.e.$$

Now, we can use convexity to obtain a stronger convergence.

Theorem 20. *Let* (Ω, μ) *be a* σ*-finite measure space and* $C \in L_p(\Omega, \mu), 1 \leq p < 2$ *be a convex Komlós set. Then, for every sequence* $\{f_n\}_n \in C$ *there exists a subsequence* $\{g_n\}_n$ *of* $\{f_n\}_n$ *and a function* $g \in C$ *such that for any subsequence* $\{h_n\}_n$ *of* $\{g_n\}_n$*,*

$$\lim_{n\to\infty} n^{-\frac{1}{p}} \sum_{k=1}^{n} (h_n - g) = 0 \ a.e.$$

Proof. Since C is convex and Komlós, it is norm bounded by Theorem 18. Now, fix $\{f_n\}_n \in C$. Without loss of generality, by taking a subsequence if necessary we may assume that there is a $g \in C$ such that for every subsequence $\{g_n\}_n$ of $\{f_n\}_n$ we have

$$\lim_{n\to\infty} n^{-1} \sum_{k=1}^{n} (g_n - g) = 0 \ a.e.,$$

since C is Komlós. Now, by Theorem 19 we have a subsequence $\{\widehat{g}_n\}_n$ of $\{f_n\}_n$ and $\widehat{g} \in L_p$ such that for any further subsequence $\{h_n\}_n$ of $\{\widehat{g}_n\}_n$

$$\lim_{n\to\infty} n^{-\frac{1}{p}} \sum_{k=1}^{n} (h_n - \widehat{g}) = 0 \ a.e.$$

This directly implies that

$$\lim_{n\to\infty} n^{-1} \sum_{k=1}^{n} (h_n - \widehat{g}) = 0 \ a.e.$$

So, $\widehat{g} = g \in C$. This concludes the proof of Theorem 20. $\square$

The restriction that (Ω, μ) be σ-finite can be removed from Theorem 20.

4.5 CLOSING REMARKS

In several theorems throughout this paper, the restriction that a Banach function space have the Fatou property can be replaced with a weaker property. Also, it might be possible to weaken or remove the FI property from Theorem 16. That would be an interesting result.

The definition of Banach function spaces varies widely. For instance, Zaanen [13] defines Banach function spaces without property (P3). This property is not needed in Theorem 16; however, it is used in the proof of Theorem 17. The definition found in [2] assumes several nice properties. There, all Banach function spaces satisfy the Fatou property, are FI, and have finite underlying measure.

APPENDIX

DELTA T MAPPING

Recall Alspach's mapping, $T : C \to C$ be defined by

$$(Tf)(x) := \operatorname{cut}(0, 1, 2f(2x))\,\chi_{E_{(0,1)}}(x) + \operatorname{cut}(1, 2, 2f(2x-1))\,\chi_{E_{(1,1)}}(x),$$

for all $x \in [0,1]$, for all $f \in C$. Now, let $\Delta : C \to C$ be defined by

$$\Delta f(t) := \begin{cases} f(2t) & , 0 \le t < \frac{1}{2} \\ 1 - f(2t-1) & , \frac{1}{2} \le t < 1. \end{cases}$$

Let $\tau_n : A_n \subset \mathbb{N} \to B_n \subset \mathbb{N}$, $A_n := \{i \in \mathbb{N}, 0 \le i < 4^n\}$ and $B_n := \{i \in \mathbb{N}, 0 \le i < 2^n\}$, be defined by

$$\tau_n(i) := \sigma_n \left(\sum_{k=0}^{n-1} 2^k d_0 \left(d_{2k}(i) + \sum_{l=1}^{k} d_{2l-1}(i) \right) \right)$$

and $\kappa : A_n \subset \mathbb{N} \to \{0,1\}$ be defined by

$$\kappa(i) := d_0 \left(\sum_{l=1}^{n} d_{2l-1}(i) \right)$$

where $\sum_{l=1}^{0} *$ is understood to be 0 and $i < 4^n$. Recall that σ_n acts on $i \in \mathbb{N}, 0 \le i < 2^n$ and is defined by $\sigma_n(i) := \sum_{j=0}^{n-1} d_{n-1-j}(i) 2^j$ where $i = \sum_{j=0}^{2n-1} d_j(i) 2^j$ with $d_j \in \{0,1\}$ $\forall j$, which is a base 2 representation of $i \in \mathbb{N}$ for any $n \in \mathbb{N}$ s.t. $i < 2^{2n}$.

Lemma 21. *For $i, j, n \in \mathbb{N}$ where n is fixed, $0 \leq i < 4^n$, and $0 \leq j < 4$;*

$$\tau_{n+1}(4^n j + i) = 2\tau_n(i) + \kappa(i) + (-1)^{\kappa(i)}\tau_1(j).$$

Later, we will need this stronger result.

Lemma 22. *For $i, j, m, n \in \mathbb{N}$ where m and n are fixed, $0 \leq i < 4^m$, and $0 \leq j < 4^n$;*

$$\tau_{n+m}(4^m j + i) = 2^n \tau_m(i) + \tau_n((4^n - 1)\kappa(i) + (-1)^{\kappa(i)} j).$$

Lemma 21 together with properties of cut lead to the following lemma.

Lemma 23. *For every $n \in \mathbb{N}$, for all $f \in C$, for all $x \in [0,1]$,*

$$(\Delta T)^n f(x) = \sum_{i=0}^{4^n-1} \left[\kappa(i) + (-1)^{\kappa(i)} cut\left(\tau_n(i), \tau_n(i) + 1, 2^n f\left(4^n x - i\right)\right)\right] \chi_{E_{(i,2n)}}(x).$$

$$\begin{aligned}(\Delta T)^n f(x) = \sum_{i=0}^{4^n-1} cut\Big(&\left(2^n - 1\right)\kappa(i) + (-1)^{\kappa(i)}\tau_n(i), \left(2^n - 1\right)\kappa(i) + (-1)^{\kappa(i)}\tau_n(i) + 1, \\ &2^n \left(\kappa(i) + (-1)^{\kappa(i)} f\left(4^n x - i\right)\right)\Big) \chi_{E_{(i,2n)}}(x).\end{aligned}$$

Lemma 24. *For any $f \in C$ and $s \in S$,*

$$\lim_{m \to \infty} \int_{[0,1]} (\Delta T)^m f \cdot s = \frac{1}{2} \int_{[0,1]} s\ .$$

This follows from Lemma 22.

Theorem 25. *$(\Delta T)^n f$ converges weakly to $\frac{1}{2}\chi_{[0,1]}$, $\forall\, f \in C$.*

Lemma 26. *For every $f \in C$, $\frac{1}{2}\chi_{[0,1]} \in D_\infty(f)$ and $D_\infty\left(\frac{1}{2}\chi_{[0,1]}\right) \subseteq D_\infty(f)$.*

Theorem 27. *$D_\infty\left(\frac{1}{2}\chi_{[0,1]}\right)$ is the only minimal invariant subset of C.*

Theorem 28. *ΔT is fixed point free on C.*

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